ECS 452: In-Class Exercise # 9

Date: 27 / 02 / 2018			
Name	ID (last 3 digits)		
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Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

1. Consider two random variables *X* and *Y* whose joint pmf matrix is given by $\mathbf{P} = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1/6 \end{bmatrix}$. Find $I(X;Y)$.

We use the formula $I(X;Y) = H(X) + H(Y) - H(X,Y)$.

$H(X,Y)$ can be found directly from the elements in the P matrix:

$$H(X,Y) = -\frac{1}{2}\log_2\frac{1}{2} - 3\times\frac{1}{6}\log_2\frac{1}{6} = \frac{1}{2}\left(-\log_2\frac{1}{2} - \log_2\frac{1}{6}\right) = \frac{1}{2}\log_2 12 \approx 1.7925$$

$H(X)$ and $H(Y)$ can be found by first finding $p(x)$ and $q(y)$ from the Q matrix:

$$P = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1/6 \end{bmatrix} \begin{matrix} \rightarrow \frac{2}{3} \\ \rightarrow \frac{1}{3} \end{matrix}$$

$\downarrow$ $2/3$ $\quad$ $\downarrow$ $1/3$

$H(X) = -\frac{2}{3}\log_2\frac{2}{3} - \frac{1}{3}\log_2\frac{1}{3} = 0.9183$

$H(Y) = 0.9183$

$q(y)$ has same probability values as $p(x)$

So, $I(X;Y) \approx 2\times 0.9183 - 1.7925$

$= 0.0441$

2. Consider two random variables *X* and *Y* whose $\underline{\mathbf{p}} = \begin{bmatrix} \frac{1}{3}, \frac{2}{3} \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} 1/6 & 5/6 \\ 2/3 & 1/3 \end{bmatrix}$. Find $I(X;Y)$.

$$\begin{matrix} \xrightarrow{\times 1/3} \\ \xrightarrow{\times 2/3} \end{matrix} \begin{bmatrix} \frac{1}{18} & \frac{5}{18} \\ \frac{4}{9} & \frac{2}{9} \end{bmatrix} = P$$

First, we find the P matrix.

Then, we follow the same steps as in question (1).

(same probability values)

$$P = \begin{bmatrix} 1/18 & 5/18 \\ 4/9 & 2/9 \end{bmatrix} \begin{matrix} \rightarrow 6/18 = 1/3 \\ \rightarrow 6/9 = 2/3 \end{matrix}$$

$\downarrow$ $9/18 = 1/2$ $\quad$ $\downarrow$ $9/18 = 1/2$

$H(X) \approx 0.9183$

$H(Y) = \log_2 2 = 1$

uniform

$$H(X,Y) = -\frac{1}{18}\log_2\frac{1}{18} - \frac{5}{18}\log_2\frac{5}{18} - \frac{4}{9}\log_2\frac{4}{9} - \frac{2}{9}\log_2\frac{2}{9} \approx 1.7472$$

Alternatively

$H(Y|X=x_1) = H([1/6 \;\; 5/6]) \approx 0.6500$

$H(Y|X=x_2) = H([2/3 \;\; 1/3]) \approx 0.9183$

$H(Y|X) = \sum_x p(x) H(Y|X=x) \approx \frac{1}{3}\times 0.6500 + \frac{2}{3}\times 0.9183$

≈ 0.8289

$I(X;Y) = H(Y) - H(Y|X) \approx 0.1711$

$I(X;Y) \approx H(X) + H(Y) - H(X,Y) = 0.9183 + 1 - 1.7472 = 0.1711$

3. (0 pt) Consider two random variables *X* and *Y* whose $\mathbf{Q} = \begin{bmatrix} 1/6 & 5/6 \\ 1/6 & 5/6 \end{bmatrix}$. Find $I(X;Y)$.

Note that the two rows in Q are identical. This means $Q(y|x)$ does not depend on x. In other words, knowing the value of X does not change the (conditional) pmf of Y. Therefore, X and Y are independent which implies $I(X;Y) = 0$.

See next page for a more direct solution.

Remark: Normally, to calculate $I(X;Y)$ you will need both $\underline{p}$ and Q.

So, there must be something special about Q that allows you to get $I(X;Y)$ without $\underline{p}$.

Direct calculation:

$H(Y|x) = H([\frac{1}{6}\ \frac{5}{6}]) \approx 0.65$ for any x.

So, $H(Y|X) = \sum_x p(x) H(Y|x) \approx 0.65 \underbrace{\sum_x p(x)}_{1} \approx 0.65$.

$I(X;Y) = H(Y) - H(Y|X)$. So, we need $H(Y)$ which in turn need $q(y)$

Let's try $p(x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \\ 0, & \text{otherwise} \end{cases}$

Then,

$$\overset{\underline{p}}{\begin{bmatrix} 1-p & p \end{bmatrix}} \overset{Q}{\begin{bmatrix} 1/6 & 5/6 \\ 1/6 & 5/6 \end{bmatrix}} = \overset{\underline{q}}{\begin{bmatrix} \frac{1}{6} & \frac{5}{6} \end{bmatrix}} \Rightarrow H(Y) = H([\tfrac{1}{6}\ \tfrac{5}{6}]) = H(Y|X)$$

regardless of the value of p

Therefore, $I(X;Y) = H(Y) - H(Y|X) = 0$.